# Kolmogorov Conditionalizers Can Be Dutch Booked (If and Only If They Are 'Evidentially Uncertain')

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**Abstract**

A vexing question in Bayesian epistemology is how an agent should update on evidence which she assigned zero prior credence. Some theorists have suggested that, in such cases, a rational agent should update by *Kolmogorov conditionalization*, according to which an agent's posterior is a regular conditional distribution in Kolmogorov's sense. What motivates this norm? Recently, Rescorla [2018] has shed light on the issue by proving a diachronic Dutch book and converse Dutch book theorem for Kolmogorov conditionalization. However, there is an important wrinkle in these results: it follows from Rescorla's converse theorem that a Kolmogorov conditionalizer is not Dutch bookable even when she plans to *always* assign *zero* posterior credence to the evidence she learns. Yet intuitively, such an agent can be Dutch booked. In this paper, we show that this phenomenon arises because Rescorla's definition of a *bookie strategy* precludes certain intuitively available Dutch books. We show that once Rescorla's definition is weakened, the converse Dutch book claim fails. We then propose a stricter norm, *Kolmogorov-Blackwell conditionalization*, which requires evidential certainty. We prove a Dutch book and converse Dutch book theorem for this new norm, and suggest that it is the more appropriate generalization of Bayesian conditionalization to these cases.

## 1 Introduction

How should an agent update her credences in light of new evidence? Orthodox Bayesianism offers the following answer:

BCOND If an agent begins with a (probabilistic) prior $P_{\text{old}}$ and acquires all and only new evidence $E$, then her posterior $P_{\text{new}}$ should be

$$P_{\text{new}}(\cdot) = P_{\text{old}}(\cdot|E) \equiv \frac{P_{\text{old}}(\cdot \cap E)}{P_{\text{old}}(E)}, \tag{1}$$

assuming $P_{\text{old}}(E) > 0$, that is, assuming $E$ had positive prior credence.

As the last clause suggests, BCOND is silent in cases where $P_{\text{old}}(E) = 0$. Yet, there are some situations in which agents acquire evidence which received zero prior credence—or at least, 'zero' according to the standard mathematical framework for modeling probabilities [Kolmogorov, 1933]. A classic example is learning the value of a continuous random variable $X$. Assuming the standard mathematical framework together with Bayesianism, $P_{\text{old}}$ can assign positive credence to at most countably many propositions of the form $X = x$. If $X$ is continuously distributed—say, $X$ represents a bus's arrival time, distributed like a bell curve—then $P_{\text{old}}(X = x) = 0$ for all $x$ even though each $X = x$ is epistemically possible. So if the agent learns $X = x$ as her evidence, BCOND does not say how she should update her credences.

The question of how to conditionalize on zero-probability propositions is a vexed issue in the foundations of probability. Some authors advocate jettisoning the standard mathematical framework and moving to a different formal apparatus for modeling probabilities [Hájek, 2003, 2011]. Others advocate a more conservative approach, which draws on the sophisticated theory of conditionalization developed by Kolmogorov [1933]. This theory of *regular conditional distributions* generalizes the *ratio formula* in (1) to an *integral formula*. Often overlooked by philosophers, it is widely used in statistical and scientific practice, and has seen renewed interest within formal epistemology [Easwaran, 2008, 2019, Gyenis et al., 2017, Gyenis and Rédei, 2017, Huttegger, 2015, 2017, Rescorla, 2015, 2018]. When applied to rational credence, it suggests the following diachronic norm:

KCOND (rough version) If an agent begins with a probabilistic prior $P_{\text{old}}$ in evidential situation $\mathcal{E}$, then she should follow an updating policy—a recipe yielding probabilistic $P_{\text{new}}$—which satisfies the integral formula with respect to $P_{\text{old}}$ and $\mathcal{E}$, provided such a policy exists.

KCOND entails BCOND as a special case, and covers a wide variety of situations, including those involving continuous random variables. But what, if anything, justifies it? Is there a way to better understand this Kolmogorov conditionalization norm?

Recently, Rescorla [2018] has shed light on this issue by proving Dutch book and converse Dutch book theorems for KCOND. These results generalize the diachronic Dutch book and converse Dutch book theorems for BCOND due to Lewis [1999] and Skyrms [1987]. Rescorla [2018] showed, roughly speaking, that if an agent violates KCOND, then a bookie can find a finite sequence of bets such that the agent views each as acceptable, and yet they together inflict a sure loss; conversely, if an agent follows KCOND, then it is impossible for a bookie to rig such a sequence against her. Even if one is skeptical that these kinds of Dutch book results can establish which diachronic norm is epistemically rational [Bacchus et al., 1990, Arntzenius, 2003, Hájek, 2008, Mahtani, 2012, Vineberg, 2016], one may still find them useful and informative when evaluating such norms.

However, there is an important wrinkle in Rescorla's results. It is well-known that BCOND entails *evidential certainty*: an agent who follows BCOND is required to assign full credence to her total evidence. This is not true of KCOND. In fact, there exist updating policies which satisfy the integral formula, but *require* the agent to always assign *zero* posterior credence to her total evidence. Thus, an agent can follow KCOND while radically violating evidential certainty. Intuitively, such updating strategies are Dutch bookable; a bookie can simply plan to buy, for whichever new $E$ the agent ends up learning, a bet on $E$. Since the agent assigns zero credence to $E$, she is willing to sell any such bet for nothing, and ends up losing with certainty. But according to Rescorla's converse Dutch book theorem, this kind of exploitation should be impossible as long as the agent's updating policy satisfies the integral formula—which it does. So there seems to be a mismatch somewhere.

In this paper, we show that the wrinkle is in Rescorla's definition of what counts as a Dutch book. In particular, Rescorla's definition of a *bookie strategy* includes a *product measurability* requirement, which ends up preventing the plan described above from counting as a Dutch book in the situation under consideration. We prove that, once the product measurability requirement is appropriately weakened, a Kolmogorov conditionalizer who radically violates evidential certainty can be Dutch booked (in this modified sense). To this extent, we suggest that KCOND is not the appropriate generalization of BCOND. Instead, Dutch book considerations point toward a stricter norm, which we call 'Kolmogorov-Blackwell conditionalization':[1]

KBCOND (rough version) An agent should follow an updating policy which not only satisfies the integral formula but also ensures 'evidential certainty'—in other words, ensures that she (almost always) assigns full credence to her total evidence—provided such a policy exists.

In the first part of the paper, after some preliminaries (§2), we give a specific example of an agent who satisfies KCOND but violates KBCOND in the radical way described (§3). We discuss Rescorla's product measurability requirement as it bears on this case, and show that once the requirement is suitably weakened, the agent can be Dutch booked as expected, and indeed so can any agent who violates KBCOND; in other words, we prove a Dutch book theorem for KBCOND (§4).

Of course, it is one thing to suggest that violations of KBCOND are irrational, and another to suggest that KBCOND is the appropriate generalization of Bayesian conditionalization. In the second part of the paper, we motivate this second suggestion by showing that an agent who satisfies KBCOND cannot be Dutch booked even according to our more permissive definition of a Dutch book. Thus, we have both a Dutch book theorem and converse Dutch book theorem for KBCOND (§5). Lastly (§6), we bring these results to bear on a well-known worry for the Kolmogorovian approach: that in many cases, the requisite kind of updating policy (one satisfying the integral formula and evidential certainty) does not exist [Seidenfeld et al., 2001].

# 2 Framework and Definitions

In what follows, we model an agent's credal space by $(\Omega, \mathcal{F})$, where $\Omega$ is the set of her epistemic possibilities, and $\mathcal{F}$ is the set of propositions she can grasp, where a proposition is conceived as a set of worlds. Formally, we assume $\mathcal{F}$ is a *sigma-field* on $\Omega$, that is, it contains $\Omega$ and is closed under complement and countable union. Typically, when $\Omega$ is finite, we assume $\mathcal{F}$ is simply the powerset of $\Omega$.

[1]The name is after David Blackwell, who drew special attention to the property of *propriety* within Kolmogorov's theory of regular conditional distributions, which in this context is equivalent to evidential certainty [Blackwell and Ryll-Nardzewski, 1963, Blackwell and Dubins, 1975].

For an agent with credal space $(\Omega, \mathcal{F})$, we follow the convention of orthodox Bayesianism and model her (prior) credences by a real-valued probability function $P : \mathcal{F} \to [0, 1]$. For the purposes of this paper, we will always assume that $P$ is countably additive. We will often use the triple $(\Omega, \mathcal{F}, P)$ to denote the agent's credal space together with her prior.

## 2.1 Updating policies

Updating policies tell an agent how to update her credences, given the different possible pieces of evidence she might receive. In order to define this notion, we first need the notion of an evidential situation.[2]

**Definition 1.** Fix a credal space $(\Omega, \mathcal{F})$. An **evidential situation** is a collection $\mathcal{E} = \{E_\omega\}_{\omega \in \Omega}$, where $E_\omega$ denotes the total evidence the agent will receive at $\omega$.

In general, $E_\omega$ could be anything—a proposition, a probability distribution, a constraint on conditional probabilities, and so on. In this paper, however, we restrict our attention to $\mathcal{E}$ with three standard additional properties:

1. Total evidence is propositional and graspable: for every $\omega \in \Omega$, $E_\omega \in \mathcal{F}$.[3]
2. Total evidence is factive: for every $\omega \in \Omega$, $\omega \in E_\omega$.
3. Total evidence is partitional: for every $\omega, \nu \in \Omega$, either $E_\omega = E_\nu$ or $E_\omega \cap E_\nu = \emptyset$. In other words, $\{E_\omega\}_{\omega \in \Omega}$ forms a partition of $\Omega$.

**Example 1** (Dice)**.** An agent is about to roll a die. Although she cannot look at the result, her friend will tell her if it landed even or odd. In this case, a simple toy model for the agent's credal space is $\Omega = \{1, 2, 3, 4, 5, 6\}$ with $\mathcal{F}$ the powerset. The agent's evidential situation is $\mathcal{E} = \{\{1, 3, 5\}, \{2, 4, 6\}\}$. (So, for instance, $E_2 = \{2, 4, 6\}$, $E_5 = \{1, 3, 5\}$.) Note that $\mathcal{E}$ satisfies all three properties.

**Example 2** (Bus)**.** An agent is waiting for a bus to arrive, and will learn the exact time $T$ it does so. In this case, we have $\mathcal{E} = \{T = t : t \geq 0\}$.

Suppose an agent starts with a prior $P$ in evidential situation $\mathcal{E}$. Naturally, an updating policy tells her which posterior credence function she should adopt given $E_\omega$, for all $\omega \in \Omega$. Following Rescorla [2018], we assume that the agent's posterior credence function is always a countably additive probability function.

[2]It should be flagged that our presentation differs slightly from Rescorla's, in that we do not begin with Kolmogorov's theory of regular conditional distributions but rather define the notion of an updating policy from scratch, similarly to [Easwaran, 2013]. Readers already familiar with Kolmogorov's theory will recognize that our KCOND-policy is a regular conditional distribution in Kolmogorov's sense, and our KBCOND-policy is an almost proper, regular conditional distribution.

[3]Thus we set aside Jeffrey-style cases that involve non-propositional evidence [Jeffrey, 1965].

**Definition 2.** Let $\Delta(\mathcal{F})$ denote the set of countably additive probabilities on $\mathcal{F}$.

**Definition 3.** Fix $(\Omega, \mathcal{F}, P)$ and an evidential situation $\mathcal{E}$. An **updating policy for** $P$ **given** $\mathcal{E}$ is a function $U$ that maps each epistemically possible world $\omega$ to a unique credence function $U_\omega \in \Delta(\mathcal{F})$, and also satisfies the following three conditions:

1. It gives the same recommendation for any two worlds that are evidentially indistinguishable according to $\mathcal{E}$, that is, if $E_\omega = E_\nu$, then $U_\omega = U_\nu$.

2. For every rational number $r$ and proposition $A \in \mathcal{F}$, the agent can grasp the proposition that her posterior credence in $A$ will be at most $r$, that is, the set $M_r = \{\omega : U_\omega(A) \leq r\}$ is a member of $\mathcal{F}$. (Equivalently: fixing $A \in \mathcal{F}$, the expected value of $U(A)$ with respect to $P$ is well-defined.)

3. For every rational number $r$, the agent can grasp the proposition that she will assign credence at most $r$ to her total evidence if she updates according to $U$, that is, the set $T_r = \{\omega : U_\omega(E_\omega) \leq r\}$ is a member of $\mathcal{F}$. (Equivalently: the expected value of $U(E) : \omega \mapsto U_\omega(E_\omega)$ with respect to $P$ is well-defined.)

Intuitively, condition 1 prevents a policy from requiring that the agent's posterior depend on information she will not obtain. Conditions 2 and 3 are less intuitive and only become relevant (assuming condition 1) when the partition $\mathcal{E}$ is uncountable. We offer two explanations for these conditions. First, an updating policy $U$ tells the agent whether she should assign at least $r$ credence to $A$, for each and every $A$ that she can grasp, based on her total evidence. It also tells the agent whether she should be at least $r$ confident in her total evidence. Thus, it seems that in order to update in accordance with $U$, the agent must be able to tell (explicitly or implicitly) whether the actual world is a member of $M_r$ and $T_r$. That is, $M_r$ and $T_r$ should be propositions that she can grasp, which is just what conditions 2 and 3 require.[4] Second, these conditions are necessary in order for the agent who updates according to $U$ to grasp whether she will end up judging, depending on what she learns, various bets as fair or favorable. Such an assumption is appropriate in the current context, since one might think that Dutch bookability is only a sign of irrationality in cases where the agent can see by her *own* lights that she will evaluate a sequence of (conditional) bets as individually acceptable even though they jointly guarantee her a sure loss.

**Example 3** (Dice, continued)**.** Suppose $P$ is uniform in the dice example. Letting $i, j \in \{1, 2, 3, 4, 5, 6\}$, the natural updating policy in situation $\mathcal{E}$ is specified by:

$$U_i(j) = \begin{cases} \frac{1}{3} & \text{when } i, j \text{ are both even or both odd,} \\ 0 & \text{otherwise.} \end{cases}$$

[4] This explanation dovetails closely the explanation that Easwaran [2013] offers for his definition of '$\mathcal{E}$-available plan' (in our terminology, an $\mathcal{E}$-available plan is a map $\omega \mapsto U_\omega(A)$ for some updating policy $U$ and proposition $A$): "In planning whether to have credence [at most] $r$ [in $A$], the agent is responding to whether the actual world is in $M_r$ or not, and it seems this responsiveness should require that $M_r$ is a proposition she can grasp."

**Example 4** (Ratio)**.** More generally, fix $(\Omega, \mathcal{F}, P)$ and $\mathcal{E}$ such that $P(E_\omega) > 0$ for all $\omega \in \Omega$. Then it is straightforward to verify that the map $U : \Omega \to \Delta(\mathcal{F})$

$$U : \omega \;\mapsto\; P(\cdot|E_\omega) \equiv \frac{P(\cdot \cap E_\omega)}{P(E_\omega)} \tag{2}$$

is an updating policy for $P$ given $\mathcal{E}$. And so we may formulate BCOND as follows:

BCOND If an agent with credal space $(\Omega, \mathcal{F})$ begins with prior $P$ in an evidential situation $\mathcal{E}$ such that $P(E_\omega) > 0$ for all $\omega \in \Omega$, then she should adopt the updating policy $U$ given by (2).

## 2.2 $\mathcal{E}$-informed propositions

In order to state the integral formula and Rescorla's product measurability requirement on a bookie strategy, we will need the notion of an *$\mathcal{E}$-informed proposition*.[5] Fix an evidential situation $\mathcal{E}$. Intuitively, a proposition is $\mathcal{E}$-informed if, for agents in evidential situation $\mathcal{E}$, it is guaranteed that after acquiring their evidence, they will be completely informed as to whether the proposition is true.

**Definition 4.** Fix a credal space $(\Omega, \mathcal{F})$ and evidential situation $\mathcal{E}$. Then a proposition $E \in \mathcal{F}$ is **$\mathcal{E}$-informed** if for all $\omega \in \Omega$, the total evidence $E_\omega$ either implies $E$ ($E_\omega \subseteq E$) or its negation ($E_\omega \subseteq E^c$). Equivalently, $E$ is $\mathcal{E}$-informed just in case there is a collection of worlds $\{\omega_i\}_{i \in I}$ such that $E = \bigcup_{i \in I} E_{\omega_i}$.

In what follows, we will abuse notation and also let $\mathcal{E} \subseteq \mathcal{F}$ denote the set of $\mathcal{E}$-informed propositions.[6] It is straightforward to show that the set $\mathcal{E} \subseteq \mathcal{F}$ of $\mathcal{E}$-informed propositions forms a sigma-field on $\Omega$.

At this point, it is worth mentioning a small but notable difference between our framework and Rescorla's. We have defined an updating policy relative to an evidential situation, which is a collection of graspable propositions that are mutually exclusive and jointly exhaustive. Rescorla, on the other hand, focuses on *Kolmogorov learning scenarios*—scenarios where "the agent gains full membership knowledge for a sub-sigma-field $\mathcal{G} \subseteq \mathcal{F}$ regarding the true outcome $\omega$." More concretely, consider the bus example (Example 2). The idea is that the agent who learns the exact time at which the bus will arrive also knows whether the bus will arrive between $a$ and $b$, for rational $a, b$. That is, she knows whether the actual world belongs to sets of the form $\{\omega : a \leq T(\omega) \leq b\}$. As a result, she "gains full membership knowledge" of $\sigma T$—the smallest sigma field generated by sets of this form. Rescorla does not elaborate on what conditions an arbitrary sub-sigma-field must meet in order for it to

[5] We borrow this terminology from Hervés-Beloso and Monteiro [2013].

[6] In other words, we will let $\mathcal{E}$ denote the entire collection of $\mathcal{E}$-informed propositions, not just the collection $\{E_\omega\}_{\omega \in \Omega}$. Note that every $E_\omega \in \mathcal{E}$ is automatically $\mathcal{E}$-informed (assuming propositionality and partitionality of evidence).

model a Kolmogorov learning scenario. We shall not pursue this question here. Instead, we remark that the learning situations that we are working with in this paper define a special class of Kolmogorov learning scenarios: those that consist of propositions informed by an evidential situation (so $\mathcal{G} = \mathcal{E}$, the sub-sigma-field of $\mathcal{E}$-informed propositions).

## 2.3 The integral formula

Recall KCOND singles out the *integral formula* as the appropriate generalization of the ratio formula.

**Definition 5.** Fix $(\Omega, \mathcal{F}, P)$ and an evidential situation $\mathcal{E}$. An updating policy $U$ satisfies the **integral formula** with respect to $P$ and $\mathcal{E}$ if, for each $A \in \mathcal{F}$ and $\mathcal{E}$-informed proposition $E$,

$$P(A \cap E) = \int_E U_\omega(A)\, dP(\omega) \tag{3}$$

The integral formula says that if the agent knows the truth-value of $E$ based on her total evidence, then for any proposition $A$, her *prior* expectation of her posterior credence in $A$, according to $U$ and restricted to $E$, should equal her prior unconditional credence in $A \cap E$. In particular, if $U$ satisfies the integral formula, then the agent's prior credence in $A$ always equals her expectation of her posterior credence in $A$ according to $U$ (in (3), this is the case where $E = \Omega$).

We now can formulate KCOND with full precision:

KCOND If an agent with credal space $(\Omega, \mathcal{F})$ begins with prior $P$ in evidential situation $\mathcal{E}$, she should adopt an updating policy $U$ that satisfies the integral formula (3) with respect to $P$ and $\mathcal{E}$.

**Example 5** (Bus, continued)**.** Suppose that the agent's prior credence for the arrival time $T$ is uniformly distributed between 0 and 1. Since $P(T = t) = 0$ for all $t$, BCOND is silent on how the agent should update her credence given $T$, but KCOND is not. For example, KCOND rules out the updating policy that tells the agent to be fully confident that the bus arrives at 1, no matter what she learns.[7]

**Example 6.** When $P(E_\omega) > 0$ for all $\omega \in \Omega$, KCOND and BCOND recommend the same updating policy, that is, the integral formula reduces to the ratio formula.[8]

[7]That is, for all $\omega$, $U_\omega$ is the Dirac measure concentrated on 1 ($U_\omega(A) = 1$ if $1 \in A$ and 0 otherwise). Since $U_\omega$ is independent of $\omega$, it trivially satisfies the three conditions for an updating policy. Now consider $A = [1/2, 1]$ and $E = \Omega = [0, 1]$. *A priori* the agent only assigns a credence of $1/2$ that the bus will arrive at a time between $1/2$ and 1 (i.e. $P(A) = 1/2$). If she updates according to $U$, however, then she will become fully confident that the bus arrives at a time between $1/2$ and 1, i.e. $\int U_w(A) dP = 1$. So $U$ does not satisfy the integral formula.

## 2.4 Bets and bookie strategies

Abstractly, a bet is a function that associates each world with a real number, representing the net payoff of the bet at that world. This function should also have a well-defined expectation, which denotes the expected payoff of the bet.

**Definition 6.** Given $(\Omega, \mathcal{F})$, a **bet** is a function $X : \Omega \to \mathbb{R}$ with well-defined expectation, i.e., for all rational $r$, the set $\{\omega : X(\omega) \geq r\}$ is a member of $\mathcal{F}$.

We say a bet is acceptable if it has non-negative expected payoff.

**Definition 7.** Given $(\Omega, \mathcal{F}, P)$, a bet $X$ is **acceptable for** $P$ if $\mathbb{E}_P[X] \geq 0$.

**Example 7** (Simple acceptable bet)**.** Fix $A \in \mathcal{F}$. The bet that costs \$$P(A)$ and pays \$1 if $A$ occurs and \$0 otherwise corresponds to the function:

$$X(\omega) = 1_A(\omega) - P(A),$$

where $1_A$ denotes the indicator:

$$1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Since $\mathbb{E}_P[X] = \mathbb{E}_P[1_A] - P(A) = P(A) - P(A) = 0$, this bet is acceptable for $P$.

A bookie strategy is a plan for what bets (if any) to offer an agent once new information becomes jointly available. In other words, a plan associates with each epistemically possible world $\omega \in \Omega$ a (possibly trivial) bet that the bookie will offer the agent at that world. We

[8] To see this, first, suppose $U$ satisfies (3). For any $\omega \in \Omega$ and $A \in \mathcal{F}$:

$$P(A \cap E_\omega) = \int_{E_\omega} U_\nu(A)\, dP(\nu) = U_\omega(A) \cdot \int_{E_\omega} dP(\nu) = U_\omega(A) \cdot P(E_\omega)$$

where the first equality follows from (3) plus the fact that $E_\omega$ is $\mathcal{E}$-informed, and the second equality follows from the first condition on an updating policy. Thus, $U_\omega(A) = \frac{P(A \cap E_\omega)}{P(E_\omega)}$ for all $A \in \mathcal{F}$, in other words, $U$ is given by (2), as desired. Conversely, suppose $U$ is given by (2), so $P(A \cap E_\omega) = U_\omega(A) \cdot P(E_\omega)$ for all $A \in \mathcal{F}$, $\omega \in \Omega$. Since $P(E_\omega) > 0$ for all $\omega$, the partition $\{E_\omega\}_{\omega \in \Omega}$ must be countable. So any $\mathcal{E}$-informed proposition can be written $E = \bigcup_{i \in I} E_{\omega_i}$ with $I$ countable. Thus, for each $A \in \mathcal{F}$ and informed $E \in \mathcal{E}$:

$$P(A \cap E) = \sum_{E_{\omega_i} \subseteq E} P(A \cap E_{\omega_i}) = \sum_{E_{\omega_i} \subseteq E} U_{\omega_i}(A) \cdot P(E_{\omega_i}) = \int_E U_\omega(A)\, dP(\omega)$$

where the first equality follows from countable additivity and the last from the definition of integration. And so $U$ satisfies the integral formula (3), as desired.

assume that the bookie has the same cognitive and experimental resources as the agent, so both share the same credal space $(\Omega, \mathcal{F})$ and are in the same evidential situation $\mathcal{E}$.[9]

We begin with our definition of a bookie strategy, and then discuss Rescorla's.

**Definition 8.** Fixing $(\Omega, \mathcal{F})$, let $\mathcal{X}$ denote the set of all bets on $(\Omega, \mathcal{F})$.

**Definition 9.** Fix $(\Omega, \mathcal{F})$ and an evidential situation $\mathcal{E}$. A **bookie strategy given** $\mathcal{E}$ is a map $S : \Omega \to \mathcal{X}$ that satisfies the following two conditions:[10]

1. $S$ recommends the same bet for two worlds that are evidentially indistinguishable to the bookie (and the agent), that is, if $E_\omega = E_\nu$, then $S_\omega = S_\nu$.

2. The bookie (and the agent) can grasp the proposition that the agent will make a net gain of at most $r$ if she accepts whatever bet the bookie offers, that is, for all rational $r$ the set $N_r = \{\omega : S_\omega(\omega) \leq r\}$ is a member of $\mathcal{F}$. (Equivalently, the function $S^* : \omega \mapsto S_\omega(\omega)$, that takes a world $\omega$ and yields the agent's net gain or loss at $\omega$ if she accepts the bookie's bet, has a well-defined expectation.)

**Example 8** (Dice, continued)**.** Recall the situation $\mathcal{E}$ in the dice example. Suppose a bookie enters and offers the agent the following conditional bet: *If the die lands odd, you pay me* \$1/3 *and I'll give you* \$1 *if it turns out the die landed on* 5*. If the die lands even, the bet is called off.* In this case, letting $A = \{5\}$, the proposition that the die lands on 5, we can model the bookie's strategy by:

$$S_\omega = \begin{cases} 1_A - 1/3 & \text{if } \omega \text{ is odd} \\ \mathbf{0} & \text{if } \omega \text{ is even,} \end{cases}$$

where $\mathbf{0}$ denotes the trivial bet, that is, the zero function. Note, for instance, that if $\omega = 5$, then the net gain is $S_\omega(\omega) = 1 - 1/3 = 2/3$.

As mentioned, Rescorla [2018] offers a different definition of a bookie strategy. In the following, $\mathcal{E}$ denotes the sigma-field of $\mathcal{E}$-informed events (§2.2).

**Definition 10.** Fix $(\Omega, \mathcal{F})$ and an evidential situation $\mathcal{E}$. A **bookie strategy in Rescorla's sense (hereafter an R-strategy) given** $\mathcal{E}$ is a map $S : \Omega \times \Omega \to \mathbb{R}$ that is measurable with

[9] This is a simplifying assumption which would not appear in a completely general theory of betting strategies, but is motivated in the current setting. We are interested in evaluating whether the agent is being rational *holding fixed* her cognitive and experimental resources. If an agent can only be Dutch booked by a bookie with more resources, it is less clear whether she exhibited a failure of rationality, or is simply being penalized for her initial disadvantage.

[10] These conditions are similar to those of an updating policy. The first condition ensures that the bookie cannot plan in advance to offer different bets at worlds that will be indistinguishable to her. For the second condition, note that when the bookie is planning what bets to offer, or when the agent is planning what bets (if any) to accept, they are responding in part to whether the actual world belongs to $N_r$ or not. This responsiveness suggests that they can grasp propositions like $N_r$.

respect to the product sigma-field $\mathcal{E} \otimes \mathcal{F}$. In particular, letting $\mathcal{E} \otimes \mathcal{F}$ denote the smallest sigma-field containing the product $\mathcal{E} \times \mathcal{F} = \{E \times A : E \in \mathcal{E}, A \in \mathcal{F}\}$, the requirement is that:

$$\forall r, \{(\omega, \nu) : S(\omega, \nu) \leq r\} \in \mathcal{E} \otimes \mathcal{F}. \tag{4}$$

It is straightforward to check that a bookie strategy in Rescorla's sense is automatically a bookie strategy in our sense. In particular, (4) implies (a)–(c):

(a) For all $\omega \in \Omega$, $S(\omega, \cdot)$ is a bet.

(b) If $E_\omega = E_\nu$, then $S(\omega, \cdot) = S(\nu, \cdot)$.

(c) For all $r$, the proposition $N_r = \{\omega : S(\omega, \omega) \leq r\}$ is a member of $\mathcal{F}$.

So, given an $R$-strategy, the map $\omega \mapsto S(\omega, \cdot)$ defines a bookie strategy $S : \Omega \to \mathcal{X}$. Crucially, however, the converse may fail to hold; $S$ might satisfy (a)–(c), but fail to satisfy (4). Thus a question arises as to whether the requirement (4) is warranted.

Rescorla (2018, 718-9, notation adapted) offers the following explanation for this product measurability requirement:

> The $\mathcal{E} \otimes \mathcal{F}$-measurability requirement may look a bit mysterious, so let me elucidate it. Suppose for purposes of this paragraph that "information" [evidence] received by the bookie may be nonveridical [non-factive]. We use $\mathcal{E} \otimes \mathcal{F}$ to model the implicit knowledge of an observer who learns what information [evidence] was transmitted to the bookie *and* learns which events in $\mathcal{F}$ occurred. Any such observer should be able in principle to determine whether the bookie's selected bet has gain $\leq r$. She should acquire implicit knowledge whether the proposition *The bet selected by the bookie has net gain* $\leq r$ is true. This proposition corresponds to the event $\{(\omega, \nu) : S(\omega, \nu) \leq r\} = S^{-1}(-\infty, r]$. $\mathcal{E} \otimes \mathcal{F}$-measurability requires that each such event belong to $\mathcal{E} \otimes \mathcal{F}$. Thus, $\mathcal{E} \otimes \mathcal{F}$-measurability requires that an observer who knows what information the bookie received *and* which events in $\mathcal{F}$ occurred is able to decide whether the bet selected by the bookie has net gain $\leq$ or $> r$.

Rescorla's explanation consists of two steps: (i) the product sigma-field $\mathcal{E} \otimes \mathcal{F}$ models "the implicit knowledge of an observer," and (ii) such an observer can in principle tell whether the bet selected by the bookie has net gain $\leq r$, and this proposition corresponds to the set $\{(\omega, \nu) : S(\omega, \nu) \leq r\}$. While Rescorla does not necessarily take this explanation to amount to an *argument* for the product measurability requirement, it is worth evaluating whether it is successful when interpreted as such. We suggest that neither claim (i) nor claim (ii) holds in the current setting.

An important background assumption of the current setting is that evidence is factive and partitional (§2.1). In Rescorla's terminology, it is common knowledge (between the agent, the bookie and the observer) that both the agent and the bookie will receive truthful information from a collection $\mathcal{E}$ of mutually exclusive and jointly exhaustive propositions. Under this assumption, if an observer learns which events in $\mathcal{F}$ occur, then she can deduce which $E \in \mathcal{E}$ the bookie receives. More precisely, if for every $A \in \mathcal{F}$, the observer knows whether $A$ is true at the actual world $\omega$ (whether $\omega \in A$), then she can deduce the actual evidence $E_\omega$, even if $\mathcal{F}$ does not contain the proposition $\{\omega\}$ that precisely describes the actual world $\omega$.[11] Thus, in this setting, an observer knows "what information was transmitted to the bookie" and "which events in $\mathcal{F}$ occurred" if and only if she knows which events in $\mathcal{F}$ occurred. Her implicit knowledge should be modeled by $\mathcal{F}$ (on $\Omega$), rather than $\mathcal{E} \otimes \mathcal{F}$ (on $\Omega \times \Omega$). To the extent that the representation $\mathcal{E} \otimes \mathcal{F}$ is still used, measurability constraints imposed on the observer using $\mathcal{E} \otimes \mathcal{F}$ should be reformulable using only $\mathcal{F}$ or $\mathcal{E}$.[12]

Turning to (ii), the question is then what corresponds to the proposition *The bet selected by the bookie has net gain* $\leq r$. Here again partitionality plays an important role: if evidence is partitional, then the world that determines the net gain of the bet selected by the bookie just is the world that determines the information according to which the bookie selects her bet in the first place. That is, the very same world (namely the actual world $\omega$) determines both the evidential basis and the eventual payoff of the bookie's betting strategy. To this extent, the proposition that the bet selected by the bookie has net gain $\leq r$ corresponds to the set:

$$\begin{aligned}&\{\omega : \text{the bet selected by the bookie at } \omega \text{ has net gain} \leq r \text{ at } \omega\}\\ &\quad = \{\omega : S(\omega, \omega) \leq r\}.\end{aligned}$$

Putting these pieces together, we see that assuming evidence is factive and partitional, what Rescorla's considerations for (i) and (ii) actually establish is that the set $\{\omega : S(\omega, \omega) \leq r\}$ belongs to $\mathcal{F}$, which is condition (c). Thus, Rescorla's explanation does not establish that we should impose the product measurability requirement (4) over and above the existing requirement (c) in the current setting.[13] Of course, this is not a positive reason to reject the product measurability requirement. But later (§3), we will see a concrete example where the requirement looks too stringent.

[11] By assumption $\mathcal{E} \subseteq \mathcal{F}$. So the observer knows whether $E_\nu$ is true at the actual world $\omega$ for every $\nu$. Since evidence is factive and partitional, it follows that $\omega \in E_\nu$ iff $E_\nu = E_\omega$, i.e. the unique $E_\nu$ that is true at the actual world $\omega$ is the actual evidence transmitted to the bookie and the agent.

[12] In another paper, Rescorla [2019] argues in favor of non-factive formulations of conditionalization, partly on the grounds that these formulations cover a wider range of cases. It would not be surprising if evidential certainty were less obligatory in this more general setting. Indeed, our Dutch book result against evidential uncertainty (§4) no longer goes through if we drop the factivity assumption that $\omega \in E_\omega$ for all $\omega \in \Omega$ (the second property of an evidential situation $\mathcal{E}$ (§2.1)), since then it no longer follows that a bet against $E_\omega$ at $\omega$ will always lose. (However, this does not mean that, in cases where the situation $\mathcal{E}$ *is* factive, evidential uncertainty should be permitted (§3).)

[13] What about in non-factive settings? Our view is that, even in those settings, one should not impose the product measurability requirement (4). The basic idea is that whether $\mathcal{E}$ is factive or non-factive should not affect what strategies are available to the bookie. And since we do not think bookie strategies must satisfy (4) in factive situations, we do not think they must do so in non-factive situations. (Note that this does not imply evidential uncertainty is also Dutch bookable in non-factive situations; see the previous footnote.) Of course, one could just as easily mount an argument in the opposite direction: since bookie strategies should satisfy (4) in non-factive situations, they should satisfy it in factive ones too. At this point in the dialectic, we point to the examples in §3 as independent reason to drop the requirement, since these examples involve strategies which seem perfectly possible to implement and satisfy (a)–(c) but violate (4).

## 2.5 Diachronic Dutch books

We would like to determine whether certain updating policies are *Dutch bookable.*

Modulo the issue of product measurability, we follow Rescorla's definition of a Dutch book almost verbatim. Roughly speaking, a *diachronic Dutch book* against $U$ is an initial bet $X$ together with a bookie strategy $S$, such that the agent views $X$ as acceptable, but if she updates according to $U$ then she will always be willing to accept or reject bets in $S$ that (together with $X$) are guaranteed to lose her money.

**Definition 11.** For an updating policy $U$ for $P$ given $\mathcal{E}$, we say a bookie strategy $S$ is **acceptable for** $U$ **at** $\omega$ if the bet $S_\omega$ is acceptable for $U_\omega$, that is, the agent's expected payoff for $S_\omega$ according to her posterior $U_\omega$ is non-negative: $\mathbb{E}_{U_\omega}[S_\omega] \geq 0$.

**Definition 12.** Fix $(\Omega, \mathcal{F}, P)$ and an evidential situation $\mathcal{E}$. Let $U$ be an updating policy for $P$ given $\mathcal{E}$. Then a **Dutch book against** $U$ is a pair $(X, S)$ such that:

1. $X$ is acceptable for $P$, that is, $\mathbb{E}_P[X] \geq 0$.

2. $S$ is a bookie strategy. Moreover, the agent can grasp the proposition that she will view the bet the bookie offers as acceptable if she updates by $U$, that is, the proposition $A = \{\omega : S_\omega \text{ is acceptable for } U \text{ at } \omega\}$ is a member of $\mathcal{F}$.

3. According to the agent, if this proposition $A$ is true, then she is guaranteed to suffer a net loss upon accepting the bookie's bet. In particular, for all epistemic possibilities $\omega$, if $S_\omega$ is acceptable for $U$ at $\omega$, then $X(\omega) + S_\omega(\omega) < 0$.

4. According to the agent, if this proposition $A$ is false, then she is guaranteed to suffer a net loss upon rejecting the bookie's bet. In particular, for all epistemic possibilities $\omega$, if $S_\omega$ is not acceptable for $U$ at $\omega$, then $X(\omega) < 0$.

We define **a Dutch book in Rescorla's sense (an R-Dutch book) against** $U$ as a Dutch book $(X, S)$ against $U$ such that $S$ is not only a bookie strategy but also an R-strategy, in that it satisfies the product measurability requirement.

**Example 9** (Bus, continued)**.** Recall the pathological updating policy $U$ from Example 5 that tells the agent to be fully confident that the bus arrives at minute 1, no matter what she learns. Consider the strategy to buy a free bet against $T < 1/2$ if the bus arrives before 1/2.

$$S_\omega = \begin{cases} -1_{T<1/2} & \text{if } T(\omega) < 1/2 \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

and a bet which pays \$1 if $T < 1/2$ is true and costs $\$P(T < 1/2)$:

$$X = 1_{T<1/2} - P(T < 1/2).$$

Then $(X, S)$ is a Dutch book against $U$.[14]

The definition above requires that the agent is guaranteed to suffer a net loss. But it is natural to wonder if this condition might be weakened to yield a kind of 'Dutch book' which is less extreme, but still rationally problematic. We follow Rescorla in also defining the notion of a *weak Dutch book*.

**Definition 13.** Fix $(\Omega, \mathcal{F}, P)$ and an evidential situation $\mathcal{E}$. Let $U$ be an updating policy $U$ for $P$ given $\mathcal{E}$. Then **weak Dutch book against** $U$ is a pair $(X, S)$ which satisfies conditions 1–2 of a regular Dutch book, as well as the following conditions:

3. The agent has positive prior that she will view the offered bet in $S$ as acceptable and then suffer a net loss upon also accepting this bet:

$$P\{\omega : S_\omega \text{ is acceptable for } U \text{ given } \omega \ \&\ X(\omega) + S_\omega(\omega) < 0\} > 0.$$

4. The agent has zero prior that she will view the offered bet in $S$ as acceptable and then benefit from a net gain upon also accepting this bet:

$$P\{\omega : S_\omega \text{ is acceptable for } U \text{ given } \omega \ \&\ X(\omega) + S_\omega(\omega) > 0\} = 0.$$

5. The agent has zero prior that she will view the offered bet in $S$ as unacceptable and then benefit from a net gain upon rejecting this bet:

$$P\{\omega : S_\omega \text{ is not acceptable for } U \text{ given } \omega \ \&\ X(\omega) > 0\} = 0.$$

We also define a **weak R-Dutch book** in the analogous way.

It turns out that if an agent is susceptible to a weak Dutch book, then it is also possible to construct a regular (strong) Dutch book against her. Thus, not much hinges on whether we choose to focus on weak Dutch books or regular Dutch books.

**Proposition 1.** *Fix $(\Omega, \mathcal{F}, P)$ and an evidential situation $\mathcal{E}$. Let $U$ be an updating policy for $P$ given $\mathcal{E}$. Then a Dutch book against $U$ exists if and only if a weak Dutch book against $U$ exists. (And similarly for R-Dutch books.)*

*Proof.* Proofs are in the Appendix. □

With this set-up in mind, we state the standard Dutch book and converse Dutch book theorems for Bayesian conditionalization, as well as the Dutch book and converse Dutch book theorem for Kolmogorov conditionalization due to Rescorla [2018].

[14] To check this, first note that $\mathbb{E}_P[X] = P(T < 1/2) - P(T < 1/2) = 0 \geq 0$. It is straightforward to check $S$ is a bookie strategy (for instance note that $S^*(\omega) = -1_{T<1/2}(\omega)$, which is $\mathcal{F}$-measurable) and that the proposition $A = \{\omega : S_\omega \text{ is acceptable for } U \text{ at } \omega\}$ is a member of $\mathcal{F}$. In fact $A = \Omega$. To see this, recall that $U_\omega$ is the Dirac measure concentrated on 1. So $\mathbb{E}_\omega[S_\omega] = 0$ if $\omega < 1/2$. On the other hand, if $\omega \geq 1/2$, then $S_\omega = 0$ and $\mathbb{E}_\omega[S_\omega] = 0$. So $S$ is acceptable at every $\omega$. Finally, note that if the bus arrives before 1/2, then the agent wins $X$ but loses $S_\omega$, and her total gain is $-\$1/2$; if the bus arrives after 1/2, then she loses $X$. So she loses money no matter what.

**Theorem 1** (Lewis [1999], Skyrms [1987]). *Fix* $(\Omega, \mathcal{F}, P)$ *and an evidential situation* $\mathcal{E}$ *such that* $P(E_\omega) > 0$ *for all* $\omega$. *Let* $U$ *be an updating policy for* $P$ *given* $\mathcal{E}$. *Then a Dutch book against* $U$ *exists if and only if* $U$ *fails to satisfy the Bayesian formula*

$$U_\omega(\cdot) = P(\cdot|E_\omega) \equiv \frac{P(\cdot \cap E_\omega)}{P(E_\omega)} \qquad \forall \omega \in \Omega.$$

**Theorem 2** (Rescorla [2018]). *Fix* $(\Omega, \mathcal{F}, P)$ *and any evidential situation* $\mathcal{E}$. *Let* $U$ *be an updating policy for* $P$ *given* $\mathcal{E}$. *Then an R-Dutch book against* $U$ *exists if and only if* $U$ *fails to satisfy the integral formula* (3) *with respect to* $P$ *and* $\mathcal{E}$.

# 3 Evidential Certainty

Evidential certainty is the idea that one should be certain of one's total evidence. Roughly speaking, an updating policy *ensures evidential certainty* if it ensures the agent following the policy will end up certain in her total evidence. More precisely:

**Definition 14.** Fix $(\Omega, \mathcal{F}, P)$ and an evidential situation $\mathcal{E}$, and let $U$ be an updating policy for $P$ given $\mathcal{E}$. Then $U$ **yields evidential certainty at** $\omega$ if $U_\omega(E_\omega) = 1$. We say $U$ **ensures evidential certainty** if it yields evidential certainty at $P$-almost every possibility $\omega$:

$$P\{\omega \in \Omega : U_\omega(E_\omega) = 1\} = 1. \tag{5}$$

Note that $U_\omega(E_\omega)$ denotes the agent's posterior credence at $\omega$ in the total evidence she acquires at $\omega$. Equation (5) says she is fully confident *ex ante* that this credence will be maximal. With this definition in hand, we can now give a precise formulation of the KBCOND norm:

KBCOND If an agent with credal space $(\Omega, \mathcal{F})$ in evidential situation $\mathcal{E}$ begins with prior $P$, she should adopt an updating policy $U$ for $P$ given $\mathcal{E}$ that both satisfies the integral formula (3) *and* ensures evidential certainty (5).

In this section, we will explore when KCOND and KBCOND can come apart. In particular, we will consider radical violations of evidential certainty which occur despite the satisfaction of the integral formula.

**Definition 15.** $U$ **maximally violates evidential certainty** if it always recommends assigning zero posterior credence to the evidence acquired:

$$U_\omega(E_\omega) = 0 \text{ for all } \omega \in \Omega. \tag{6}$$

## 3.1 Evidential certainty and the integral formula

In many situations, the integral formula does guarantee evidential certainty:

**Definition 16.** Fix $(\Omega, \mathcal{F})$. An evidential situation $\mathcal{E}$ is **quasi-separable** if there exists a sub-sigma-field $\mathcal{G} \subseteq \mathcal{E}$ of $\mathcal{E}$-informed propositions which is countably generated and contains $E_\omega$ for all $\omega \in \Omega$.

**Proposition 2** (Generalization of Seidenfeld et al. [2001], Theorem 1)**.** *Fix* $(\Omega, \mathcal{F}, P)$ *and* $\mathcal{E}$*. If* $\mathcal{E}$ *is quasi-separable, then* $U$ *ensures evidential certainty if it satisfies the integral formula, and so* KCOND *and* KBCOND *align.*

One consequence of this theorem is that if $X : \Omega \to \mathbb{R}$ is a random variable and $\mathcal{E} = \{X = x : x \in \mathbb{R}\}$ (such as in the bus example), KCOND and KBCOND will always give the same advice. That is because $\sigma X \subseteq \mathcal{E}$—the smallest sigma-field containing the $\mathcal{E}$-informed proposition $X \geq r$ for every rational $r$—is countably generated and contains $X = x$ for all $x$.

However, when $\mathcal{E}$ is not quasi-separable, KCOND and KBCOND can come apart, as we see next.

## 3.2 Radical evidential uncertainty: An example

The following example is adapted from a similar case in [Seidenfeld et al., 2001].

As a warm-up to the main example, we consider a simple finite case. Suppose there is a collection of $N$ identical particles ('identical' in the sense of 'indistinguishable' from particle physics). These particles are going to be fed through magnets, and either go down (0) or up (1), with equal probability.

One model for this scenario is as follows. Let $\mathbf{S} = \{0, 1\}^N$ the possible up-down results, and $\mathcal{S}$ the powerset of $\mathbf{S}$. Then we can consider the space $(\mathbf{S}, \mathcal{S}, \mu)$, where $\mu$ is the 'fair coin' measure. Note that $\mu$ has a special property: it respects the indistinguishability of the particles, in the sense that if $\pi : \mathbf{S} \to \mathbf{S}$ is a permutation of the particles (the coordinates of $\mathbf{S}$), then $\mu(\pi^{-1}S) = \mu(S)$ for all $S \in \mathcal{S}$. In other words, $\mu$ is a *symmetric probability*.

However, a problem with this model is that, even though $\mu$ does not distinguish possibilities that should be identified, the background space $\mathbf{S}$ does. For example, $(1, 0, ..., 0)$ and $(0, 1, ..., 0)$ denote different elements, even though physically they represent the same state of affairs. In light of this problem, a natural move is to *quotient out* the space $\mathbf{S}$ by identifying possibilities that are related by a permutation. That is, we say $s_1 \sim s_2$ iff there exists a permutation $\pi$ on $S$ such that $\pi(s_1) = s_2$. Clearly $\sim$ is an equivalence relation. Let $\tilde{\mathbf{S}} = \mathbf{S}/\sim$ and $\varphi : \mathbf{S} \to \tilde{\mathbf{S}}$ the quotient map. This map induces a quotient probability space $(\tilde{\mathbf{S}}, \tilde{\mathcal{S}}, \tilde{\mu})$. Here we can view $\tilde{\mathcal{S}}$ as the result of taking the set $\mathcal{G} \subseteq \mathcal{S}$ of *symmetric events*,

$$\mathcal{G} = \{S \in \mathcal{S} : \forall \pi, \, \pi^{-1}S = S\},$$

and passing it through $\varphi$, so $\tilde{\mathcal{S}} = \varphi[\mathcal{G}] \equiv \{\varphi[G] : G \in \mathcal{G}\}$. Naturally, we define $\tilde{\mu}$ by the relation $\mu(G) = \tilde{\mu}(\varphi[G])$ for all $G \in \mathcal{G}$.

**Example 10** (Identical particles, finite)**.** Let $(\Omega, \mathcal{F}, P) = (\tilde{\mathbf{S}}, \tilde{\mathcal{S}}, \tilde{\mu})$ as described above. Set $N = 2$. Consider the evidential situation $\mathcal{E}$ in which the agent learns exactly which world she is in: $\mathcal{E} = \{\{\omega\} : \omega \in \Omega\} = \{\{(0, 0)\}, \{(1, 0), (0, 1)\}, \{(1, 1)\}\}$. Then the intuitive updating policy for $P$ given $\mathcal{E}$ is specified by,

$$U : \omega \mapsto \delta_\omega(\cdot),$$

where $\delta_\omega$ is the Dirac measure concentrated at $\omega$: for all $A \in \mathcal{F}$:

$$\delta_\omega(A) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Suppose $P\{\omega\} > 0$ for all $\omega$. Then this is the only updating policy which ensures evidential certainty. It is also the only updating policy that is consistent with the integral formula. Indeed, note that in such a case, $\delta_\omega(\cdot) = \frac{P(\cdot \cap \{\omega\})}{P\{\omega\}}$.

Next, we consider the scenario where there are arbitrarily many identical particles, so $\mathbf{S} = \{0,1\}^{\aleph_0}$, $\mathcal{S}$ is the sigma-field generated by sets of outcomes that agree on finitely many particles, and $\mu$ is the 'fair coin' measure. Suppose we identify possibilities which differ by a finite permutation $\pi$.[15] Then we obtain $(\tilde{\mathbf{S}}, \tilde{\mathcal{S}}, \tilde{\mu})$, where again $\tilde{\mathcal{S}}$ is the image $\varphi[\mathcal{G}]$ of the algebra $\mathcal{G} \subseteq \mathcal{S}$ of symmetric events.

**Example 11** (Identical particles)**.** Let $(\Omega, \mathcal{F}, P) = (\tilde{\mathbf{S}}, \tilde{\mathcal{S}}, \tilde{\mu})$. Consider the evidential situation $\mathcal{E}$ where the agent will learn exactly what world she is in, that is, $\mathcal{E} = \{\{\omega\} : \omega \in \Omega\}$. Then the intuitive updating policy for $P$ given $\mathcal{E}$ is:

$$U : \omega \mapsto \delta_\omega(\cdot).$$

One can show $U$ is indeed an updating policy and satisfies the integral formula. Clearly, $U$ ensures evidential certainty, since

$$U_\omega(E_\omega) = U_\omega\{\omega\} = \delta_\omega\{\omega\} = 1.$$

However, in this infinite case, there are also policies which satisfy the integral formula but do not ensure evidential certainty. In fact, it turns out that the trivial updating policy which tells the agent to ignore her evidence,

$$U' : \omega \mapsto P(\cdot),$$

satisfies the integral formula. However, this trivial policy maximally violates evidential uncertainty:

$$U'_\omega(E_\omega) = P(E_\omega) = 0 \text{ for all } \omega \in \Omega.$$

**Proposition 3.** *Fix $(\Omega, \mathcal{F}, P) = (\tilde{\mathbf{S}}, \tilde{\mathcal{S}}, \tilde{\mu})$ as specified, with $\mu$ the fair coin measure, and $\mathcal{E} = \{\omega : \omega \in \Omega\}$. Then $U : \omega \mapsto \delta_\omega(\cdot)$ is an updating policy for $P$ given $\mathcal{E}$ that satisfies the integral formula and ensures evidential certainty. However, there also exists an updating policy $U'$ for $P$ given $\mathcal{E}$ which satisfies the integral formula but maximally violates evidential*

[15] The example is loosely inspired by the Bose-Einstein symmetry [Hudson and Moody, 1976].

*certainty, namely the trivial updating policy $U' : \omega \mapsto P(\cdot)$. More generally, if $\mu$ is symmetric (not necessarily fair), then there is an updating policy $U'$ for $P$ given $\mathcal{E}$ which satisfies the integral formula but yields $U'_\omega(E_\omega) = 0$ for all $\omega \neq \varphi(0, 0, 0, ...), \varphi(1, 1, 1, ...)$.*

Consider an agent who adopts the updating policy $U'$ in this scenario. We submit that this agent is not being rational. KBCOND recovers this intuitive verdict: the agent adopted an updating policy which violates evidential certainty even though an alternative satisfying both evidential certainty and the integral formula was available. However, since $U'$ satisfies the integral formula, KCOND judges nothing amiss.

We take this example to provide good independent motivation for rejecting the norm of Kolmogorov conditionalization in favor of something stricter. But in addition, this example opens the door to a kind of Dutch book worry against Kolmogorov conditionalization.

### 3.3 Radical evidential uncertainty and (R-)Dutch books

Intuitively, radical violations of evidential certainty should be Dutch bookable. Here is a simple bookie strategy: at each world $\omega \in \Omega$, buy a free bet from the agent that pays \$1 if (and only if) the total evidence obtained at that world, $E_\omega$, is true. It is easy to check that this plan, represented formally by the map,

$$S : \omega \mapsto -1_{E_\omega}, \tag{7}$$

constitutes a bookie strategy given $\mathcal{E}$ that is almost always acceptable (since an agent who radically violates evidential certainty will be certain that her total evidence is false). Indeed, it is easy to check that the pair $(X, S)$, where $X = \mathbf{0}$ is the trivial bet and $S$ is the strategy given by (7), constitutes a Dutch book against any $U$ that maximally violates evidential certainty.

However, while radical violations of evidential certainty can be Dutch booked in our sense of 'Dutch book', they cannot always be Dutch booked in Rescorla's sense. The reason is that while (7) always consitutes a bookie strategy, it does not always constitute an R-strategy. In other words, it does not always satisfy the product measurability requirement. We now provide a necessary and sufficient condition for (7) to count as an R-strategy, and show that this condition fails in the example above. Thus, the example does not contradict Rescorla's converse Dutch book theorem.

**Theorem 3.** *Fix any $(\Omega, \mathcal{F})$ and evidential situation $\mathcal{E}$. For any bookie strategy $S$ given $\mathcal{E}$ that can be expressed in the form (7), $S$ is an R-strategy given $\mathcal{E}$ if and only if $\mathcal{E}$ is quasi-separable.*[16]

**Proposition 4.** *In this particle example, $\mathcal{E}$ is not quasi-separable.*

## 4 Dutch Book Theorem for KBCOND

We now show more generally that any updating policy which fails to ensure evidential certainty is susceptible to a Dutch book in our sense. Coupled with the finding [Rescorla,

[16] The idea for this theorem is based on a similar result and proof due to Musiał [1980].

2018] that if an updating policy violates the integral formula it is susceptible to an R-Dutch book (and hence to a Dutch book), this result provides us with a Dutch book theorem for Kolmogorov-Blackwell conditionalization.

**Theorem 4** (Dutch book theorem)**.** *Fix $(\Omega, \mathcal{F}, P)$ and any evidential situation $\mathcal{E}$. Let $U$ be an updating policy for $P$ given $\mathcal{E}$. If $U$ violates the requirements of* KBCOND*, then a Dutch book against $U$ exists. In particular, if $U$ violates the integral formula or fails to ensure evidential certainty, then a Dutch book against $U$ exists.*

The idea behind the proof (see Appendix) is to consider the set $G$ on which evidential certainty is violated:

$$G = \{\omega : U_\omega(E_\omega) < 1\}.$$

The bookie can offer the following conditional bet on $G$: *For all $\omega$, if $G$ obtains, you pay me \$$(1 - U_\omega(E_\omega))$ and then I'll pay you \$1 iff $E_\omega$ is false. If $G$ does not obtain, no money changes hands.* Formally, this plan can be represented by the map:

$$S_\omega = \begin{cases} U_\omega(E_\omega) - 1_{E_\omega} & \text{if } \omega \in G \\ \mathbf{0} & \text{if } \omega \notin G. \end{cases}$$

It is straightforward to show that $S$ is always a bookie strategy (however, by Theorem 3, it is an R-strategy if and only if $\mathcal{E}$ is quasi-separable). It is also straightforward to show that if $P(G) > 0$, then $(\mathbf{0}, S)$ is a weak Dutch book against $U$. By the reasoning of Proposition 1, it then follows that there exists a choice of $X$ such that $(X, S)$ is a regular (strong) Dutch book against $U$.

# 5 Converse Dutch Book Theorem for KBCOND

The previous result shows that Kolmogorov conditionalizers who violate KBCOND can be Dutch booked. But it leaves open the possibility that KBCOND, like KCOND, is not strict enough. Is satisfying evidential certainty, on top of the integral formula, sufficient to avoid Dutch books?

Fortunately, as the next result shows, even on our weaker definition of a Dutch book and bookie strategy, as long as an agent follows an updating policy that satisfies both the integral formula and evidential certainty, she cannot be Dutch booked.

**Lemma 1.** *Let $U$ be an updating policy for $P$ given $\mathcal{E}$ and $S$ a bookie strategy given $\mathcal{E}$. Then the set $\{\omega : U_\omega(E_\omega) = 1\}$ belongs to the sub-sigma-field $\mathcal{E} \subseteq \mathcal{F}$.*

**Lemma 2.** *Let $U$ be an updating policy for $P$ given $\mathcal{E}$ and $S$ a bookie strategy given $\mathcal{E}$. If $U$ yields evidentially certainty at $\omega$, then $\mathbb{E}_{U_\omega}[S_\omega] = \mathbb{E}_{U_\omega}[S^*]$. Also, the set $\{\omega : \mathbb{E}_{U_\omega}[S^*] \geq 0\}$ belongs to the sub-sigma-field $\mathcal{E} \subseteq \mathcal{F}$.*

**Lemma 3** (Adapted from Çınlar [2011] Proposition IV.2.4.)**.** *Suppose $U$ is an updating policy for $P$ given $\mathcal{E}$ which respects the integral formula. Then for all random variables $X \in \mathcal{F}$, the map $U[X] : \omega \mapsto \mathbb{E}_{U_\omega}[X]$ satisfies the generalized integral formula, in that for all random variables $Y \in \mathcal{E} \subseteq \mathcal{F}$, $\mathbb{E}_P[X \cdot Y] = \mathbb{E}_P[U[X] \cdot Y]$.*

**Theorem 5** (Converse Dutch book theorem). *Fix $(\Omega, \mathcal{F}, P)$ and any evidential situation $\mathcal{E}$. Let $U$ be an updating policy for $P$ given $\mathcal{E}$. If $U$ satisfies the integral formula and ensures evidential certainty, then no Dutch book against $U$ exists.*

For example, this theorem implies that the more reasonable updating policy from the particle case (Example 11), namely $U : \omega \to \delta_\omega(\cdot)$, is immune from Dutch books. To get a sense of the reasoning behind the theorem, we can consider what would go wrong if such a policy were indeed Dutch bookable. Suppose toward contradiction there exists a pair $(X, S)$ constituting a Dutch book against $U$. Consider the set:

$$A = \{\omega : \mathbb{E}_{U_\omega}[S_\omega] \geq 0\}.$$

By definition, since $(X, S)$ is a strong Dutch book, we have $X + S^* < 0$ on $A$ and $X < 0$ on $A^c$. Note that $\mathbb{E}_{U_\omega}[S_\omega] \geq 0$ if and only if $S^*(\omega) \geq 0$ since $U_\omega$ is simply the Dirac measure concentrated at $\omega$. It therefore follows that $S^* \geq 0$ on $A$ and hence $X < 0$ on $A$. However, this means that $X < 0$ on both $A$ and $A^c$, which contradicts the assumption that $X$ is acceptable, i.e. $\mathbb{E}_P[X] \geq 0$.

# 6 Dutch Books and Non-Existence

Both KCOND and KBCOND contain a crucial caveat: 'provided such a policy exist'. The caveat is crucial because in some situations, updating policies $U$ satisfying the integral formula, or both the integral formula and evidential certainty, do not exist. It would not be appropriate to rationally require the agent to do the impossible, hence the norms KCOND and KBCOND simply go silent in such situations.

This non-existence issue looks especially worrying for KBCOND. As Seidenfeld et al. [2001] show, there are several cases where although updating policies satisfying the integral formula do exist, they all radically violate evidential certainty.[17]

It is natural to wonder whether the Dutch book results discussed above might shed some light on this issue. Indeed, Rescorla [2018] notices a potential insight these results provide in the case of KCOND. Why does KCOND go silent when it does? Because, as Rescorla's Dutch book theorem shows, these are cases where all options are problematic: all policies the agent might follow are Dutch bookable.

With Theorem 4 in hand, we can extend this insight to KBCOND. Why does KBCOND go silent when it does? Because, as the theorem shows, these are cases where all options are problematic: all policies are Dutch bookable.

Now, as Rescorla also emphasizes, there is an important caveat here. We have been restricting attention to updating policies $U$ which preserve countable additivity, that is, $U_\omega \in \Delta(\mathcal{F})$ for all $\omega$. But if we expand to allow updating policies which are merely finitely additive, then the scope of 'all options' becomes much wider, and may include some options that are

[17] For instance [Seidenfeld et al., 2001, Theorem 5], consider the particle example, except this time let $(\Omega, \mathcal{F}, P) = (S, \mathcal{S}, \mu)$. Let $\mathcal{E} = \{\Pi(\omega) : \omega \in \Omega\}$ where $\Pi(\omega)$ denotes the set of strings that are related to $\omega$ by a finite permutation. Then there exist updating policies $U$ which satisfy the integral formula with respect to $P$ and $\mathcal{E}$, but they all radically violate evidential certainty.

not so problematic [Dubins, 1975, 1977, Dubins and Heath, 1983, Maitra and Ramakrishnan, 1988, Berti and Rigo, 1999, 2002].[18]

However, while the Dutch book results don't completely solve the worry, they shed light on the debate: *conditional* on countable additivity as a background assumption, there is a good reason for this silence or non-existence. Of course, the question then becomes whether this background assumption is justified. Some might simply take the upshot of these cases to be that it should be rejected.

# 7 Conclusion

In this paper, we have argued that Kolmogorov conditionalizers can be Dutch booked. We have proposed a stricter norm, of *Kolmogorov-Blackwell conditionalization*, and proved a Dutch book (§4) and converse Dutch book theorem (§5) for this norm.

We take this discussion to have three main upshots. The first is the formulation and motivation of a new diachronic norm for orthodox Bayesianism. As we saw (§3), it is important to draw attention to this norm, since an agent who follows BCOND and KCOND might still violate KBCOND in a radical way.

Secondly, the discussion further highlights the delicateness of defining what counts as a problematic 'Dutch book' in infinitary settings. This has been observed in several ways in the literature. For example, it is well-known that if one allows for infinite bets, the prospects for rigging a Dutch book multiply [McGee, 1994, Arntzenius et al., 2004], or if one does not require the set of possibilities which yield a net loss to have positive credence, then any agent who assigns credence 0 to a non-empty proposition can be Dutch booked [Shimony, 1955, Hájek, 2008]. Here, we have drawn attention to another way the choice of definition is a subtle issue, namely the choice of measurability requirements. As we saw (§2.4, §3.3), different requirements on what propositions must be graspable to the bookie or the agent can have a significant effect on which updating policies are Dutch bookable.

Finally, we hope that the discussion has built upon the insight of [Rescorla, 2018] that the notion of a diachronic Dutch book can be defined for generalized conditioning (§2) and that these Dutch book considerations may have significance for issues like non-existence (§6). Regarding future work, we leave it open whether these considerations can also be extended to generalized Jeffrey conditionalization, where $\mathcal{E}$ is not propositional [Armendt, 1980]. We also leave it open whether other kinds of considerations, such as those based on accuracy [Leitgeb and Pettigrew, 2010, Easwaran, 2013, Briggs and Pettigrew, 2018], can be brought to bear on Kolmogorov-Blackwell conditionalization.

[18] In some cases, these options look like they include policies which are not, modulo violations of countable additivity, pathological or Dutch bookable. (Note that our definition of 'Dutch book' requires the sequence of bets is finite, so the standard Dutch book argument for countable additivity does not apply. The assumption is made to avoid the controversies associated with infinite Dutch books (Seidenfeld and Schervish, 1983; Arntzenius et al., 2004; Rescorla, 2018, fn. 9).)

# A Proofs

## A.1 Propositions

*Proof.* (of Proposition 1) The non-trivial direction is ($\Leftarrow$). Looking at the definition of a weak Dutch book, let $C, D, E$ denote the propositions (sets) from clauses $3, 4, 5$ respectively. Suppose a weak Dutch book $(X, S)$ against $U$ exists. Define a bet $Z$ by

$$\begin{aligned} Z(\omega) =& [(P(C) - 1)(X(\omega) + S_\omega(\omega))] \cdot 1_C(\omega) + L \cdot 1_{C^c}(\omega) \\ &- (X(\omega) + S_\omega(\omega)) \cdot 1_D(\omega) - X(\omega) \cdot 1_E(\omega). \end{aligned}$$

where

$$L = \min\left(\int_C (X(\omega) + S_\omega(\omega))\, dP(\omega), -1\right).$$

We verify that $(X + Z, S)$ is a strong Dutch book. For the first condition, note that

$$\begin{aligned} \mathbb{E}_P[X + Z] &= \mathbb{E}_P[X] + \mathbb{E}_P[[(P(C) - 1)(X + S^*)] \cdot 1_C] + \mathbb{E}_P[L \cdot 1_{C^c}] \\ &= \mathbb{E}_P[X] + (P(C) - 1) \cdot \int_C (X(\omega) + S_\omega(\omega))\, dP(\omega) - (P(C) - 1) \cdot L \\ &\geq 0 \end{aligned}$$

using the first inequality follows from the linearity of $\mathbb{E}_P$ and the assumption that $\mathbb{E}_P[1_D] = 0$ and $\mathbb{E}_P[1_E] = 0$, the second equality follows from the fact $P(C^c) = -(P(C) - 1)$, and the final equality follows from the assumption $\mathbb{E}_P[X] \geq 0$ together with the definition of $L$ and the fact that $X(\omega) + S_\omega(\omega) < 0$ on $C$. The second condition is immediately satisfied, since the bookie strategy $S$ is unchanged. For the third condition, suppose $S_\omega$ is acceptable for $U$ at $\omega$. If $X(\omega) + S_\omega(\omega) < 0$ then $\omega \in C$ and

$$\begin{aligned} (X + Z)(\omega) + S_\omega(\omega) &= [(P(C) - 1)(X(\omega) + S_\omega(\omega))] + X(\omega) + S_\omega(\omega) \\ &= P(C)(X(\omega) + S_\omega(\omega)) < 0 \end{aligned}$$

using the fact that $C, D, E$ are disjoint. If $X(\omega) + S_\omega(\omega) > 0$ then $\omega \in D$ and

$$\begin{aligned} (X + Z)(\omega) + S_\omega(\omega) &= L - (X(\omega) + S_\omega(\omega)) + X(\omega) + S_\omega(\omega) \\ &= L < 0. \end{aligned}$$

If $X(\omega) + S_\omega(\omega) = 0$ then $\omega$ is not a member of $C, D$ or $E$, and so

$$(X + Z)(\omega) + S_\omega(\omega) = L + X(\omega) + S_\omega(\omega) = L < 0.$$

For the fourth condition, suppose $S_\omega$ is not acceptable for $U$ at $\omega$ (and so $\omega \notin C, D$). If $X(\omega) > 0$ then $\omega \in E$ and

$$(X + Z)(\omega) = X(\omega) + L - X(\omega) = L < 0.$$

If $X(\omega) < 0$ then $\omega \notin C, D, E$ and so $(X + Z)(\omega) = L < 0$. And we are done. Similar reasoning holds for R-Dutch books and weak R-Dutch books respectively. □

*Proof.* (of Proposition 2) Let $\{G_i\}_{i\in\mathbb{N}}$ be the generators for $\mathcal{G} \subseteq \mathcal{E}$. Adapting a standard result for conditional expectations (see [Seidenfeld et al., 2001, Lemma 1]), note that if $U$ satisfies the integral formula, then for each $i$,

$$P\{\omega : U_\omega(G_i) = 1_{G_i}\} = 1.$$

Since the $G_i$ are countable, $D = \{\omega : U_\omega(G_i) = 1_{G_i} \text{ for all } i\}$ also has measure one. We now consider the $\pi$-system $\mathcal{C}$ formed by finite intersections of the $G_i$ (note that $\mathcal{G} = \sigma(\mathcal{C})$), and also

$$\mathcal{D} = \{G \in \mathcal{G} : U_\omega(G) = 1_G(\omega) \text{ for all } \omega \in D\}.$$

It is easy to check, using the fact that $U_\omega$ is a countably additive probability measure, that $\mathcal{D}$ is a $\lambda$-system, also known as a Dynkin or d-system. By the Dynkin $\pi$-$\lambda$ theorem, since $\mathcal{C} \subseteq \mathcal{D}$ we have $\mathcal{G} \subseteq \mathcal{D}$. Since $E_\omega \in \mathcal{G}$ for all $\omega$ by hypothesis, it follows that $U_\omega(E_\omega) = 1_{E_\omega}(\omega) = 1$ for all $\omega \in D$. That is, $U$ yields evidential certainty at all $\omega$ in the probability-one set $D$. □

*Proof.* (of Proposition 3) We split the proof into two parts, part (i) where we assume $\mu$ is the fair coin measure and part (ii) where we only assume $\mu$ is symmetric.

(i) First we show that $U : \omega \mapsto \delta_\omega(\cdot)$ is a KBCOND-policy. Note that $E_\omega = E_{\omega'}$ only if $\omega = \omega'$ in which case $U_\omega = U_{\omega'}$. So $U$ does not distinguish evidentially indistinguishable possibilities. Next, fix $A \in \mathcal{F}$. Note that $\{\omega : U_\omega(A) \leq r\} = \{\omega : 1_A(\omega) \leq r\}$, which clearly belongs to $\mathcal{F}$. Finally note that $\{\omega : \delta_\omega\{\omega\} \leq r\}$ belongs to $\mathcal{F}$ automatically since $\delta_\omega\{\omega\}$ always equals 1. To check $U$ respects the integral formula, note that every $A \in \mathcal{F}$ is $\mathcal{E}$-informed and note that for all $A, B \in \mathcal{F}$, we have:

$$\int_B U_\omega(A)\, dP(\omega) = \int_B \delta_\omega(A)\, dP(\omega) = \int_B 1_A(\omega)\, dP(\omega) = P(A \cap B)$$

as desired. Finally, to see $U$ satisfies evidential certainty, note that $U_\omega(\{\omega\}) = \delta_\omega(\{\omega\}) = 1$ for all $\omega \in \omega$.

Next consider the updating policy $U'$ given by $U'_\omega(A) = P(A)$. Intuitively $U'$ tells the agent to ignore her evidence and retain her prior credence function. We claim that $U'$ is a KCOND-policy. By the Hewitt-Savage 0-1 law [Hewitt and Savage, 1955], for every $A \in \mathcal{F}$, $P(A) \in \{0, 1\}$. Therefore for every $A, B \in \mathcal{F}$, $P(A \cap B) = P(A)P(B) = \int_B P(A)dP(\omega) = \int_B U'_\omega(A)dP(\omega)$. So $U'$ satisfies the integral formula. However, note that for every infinite binary sequence $s \in \mathbf{S}$, there are only countably many sequences that are related to $s$ by a finite permutation. So for every $s$, $\mu\{s' : s' \sim s\} = 0$. Since each $\omega \in \Omega$ corresponds to the equivalence class of a unique $s$, and $P\{\omega\} = \mu\{s' : s' \sim s\}$, it follows that $P\{\omega\} = 0$. Equivalently, $U'_\omega(E_\omega) = 0$, and so $U'$ radically violates evidential certainty, as claimed.

(ii) By the same reasoning as in part (i), $U : \omega \mapsto \delta_\omega(\cdot)$ is a KBCOND-policy. Next, we show there exists a $U'$ as described in the proposition. Applying the result due to Hewitt and Savage [1955] (see also [Seidenfeld et al., 2001, 1618-9]), since $\mu$ is symmetric it can be expressed as an average of extreme symmetric probabilities of the form

$$\mu(\cdot) = \int_\Theta \mu_\theta(\cdot)\, d\alpha(\theta)$$

where $\Theta = [0, 1]$, $\mu_\theta(\cdot)$ is the i.i.d. binomial product probability on $\mathcal{S}$ with $\mu_\theta(1 \times \{0,1\} \times \{0,1\} \times ...) = \theta$ and $\alpha$ is some probability on the Borel subsets of $\Theta$. Thus, for any $G \in \mathcal{G}$, we have (where $\tilde{G} \equiv \varphi(G)$):

$$\tilde{\mu}(\tilde{G}) = \mu(G) = \int_\Theta \mu_\theta(G)\, d\alpha(\theta) = \int_\Theta \tilde{\mu_\theta}(\tilde{G})\ d\alpha(\theta),$$

and so, since $\mathcal{F} = \tilde{S} = \varphi[\mathcal{G}]$, it follows that

$$P(\cdot) \equiv \tilde{\mu}(\cdot) = \int_\Theta \tilde{\mu_\theta}(\cdot)\, d\alpha(\theta). \tag{8}$$

We first note that for every $F \in \mathcal{F}$ and $\theta \in \Theta$, $\tilde{\mu_\theta}$ satisfies the integral formula with respect to $\tilde{\mu_\theta}$ and $\mathcal{E} = \mathcal{F}$, in the sense that

$$\tilde{\mu_\theta}(F \cap F') = \int_{F'} \tilde{\mu_\theta}(F)\, d\tilde{\mu_\theta}(\omega) \qquad \text{for all } F' \in \mathcal{F}, \tag{9}$$

since $\tilde{\mu_\theta}$ is extreme-valued on $\mathcal{F}$ (which holds, in turn, because $\mu_\theta$ is extreme-valued on $\mathcal{S}$). Applying a lemma due to [Seidenfeld et al., 2001, 2006, Lemma 3], it follows from (8) and (9), plus the fact that $\tilde{\mu_\theta}$ is countably additive, that for each $\omega \in \Omega$ there exists a probability $\beta_\omega$ on the Borel subsets of $\Theta$ such that for all $F \in \mathcal{F}$, the map

$$P_\mathcal{F} : F \times \omega \mapsto \int_\Theta \tilde{\mu_\theta}(F)\, d\beta_\omega(\theta)$$

is measurable with respect to $\mathcal{F}$ and satisfies the integral formula with respect to $P$ and $\mathcal{E} = \mathcal{F}$, as well as the constraint that $P_\mathcal{F}(\cdot, \omega)$ is a countably additive probability measure for every $\omega \in \Omega$; in other words, $P_\mathcal{F}$ is a regular conditional distribution for $P$ given $\mathcal{F}$ [Rescorla, 2018, Easwaran, 2019]. Defining $U'$ as the map

$$U' : \omega \mapsto P_\mathcal{F}(\cdot, \omega)$$

it follows that each $U'_\omega$ is countably additive and that $U'$ satisfies the integral formula with respect to $P$ and $\mathcal{F}$:

$$P(F \cap F') = \int_{F'} U'_\omega(F)\, dP(\omega) \qquad \text{for all } F' \in \mathcal{F}.$$

Thus, to complete the proof, it suffices to show that $U'$ satisfies the conditions of an updating policy, and assigns $U_\omega(E_\omega) = 0$ for all $\omega \neq \varphi(0,0,...), \varphi(1,1,...)$.

The first two conditions of an updating policy follow from the fact that $U_\omega = P_\mathcal{F}(\cdot, \omega)$ where $P_\mathcal{F}$ is an rcd. It only remains to check the third condition. In particular, we must show $T_r = \{\omega : U'_\omega\{\omega\} \leq r\} \in \mathcal{F}$. This follows from the result below that $U'_\omega\{\omega\} = 0$ for all $\omega$ except perhaps on the finite number of (measurable) special points $\varphi(0,0,...)$ and $\varphi(1,1,...)$.

Note that if $s \neq (0,0,...), (1,1,...)$, then $G = \Pi(s)$, the set of states which differ from $s$ by a finite permutation, is countably infinite. So, for all $\theta$, we have $\mu_\theta(G) = \sum_{i=1,2,..} \mu_\theta\{\omega_i\} = \sum_{i=1,2,..} 0 = 0$ using the fact that $\mu_\theta(s) = 0$ for all $s \neq (0,0,...), (1,1,...)$ regardless of the value of $\theta$. Thus for all $s \neq (0,0,...),(1,1,...)$, $\tilde{\mu_\theta}\{\varphi(s)\} = 0$. Since every $\omega \in \Omega$ can be expressed as $\varphi(s)$ for some $s \in S$, this implies that

$$U'_\omega\{\omega\} = \int_\Theta \tilde{\mu_\theta}\{\omega\}\, d\beta_\omega(\theta) = \int_\Theta \mathbf{0}\, d\beta_\omega(\theta) = 0$$

for all $\omega \neq \varphi(0,0,...), \varphi(1,1,...)$ as desired. □

*Proof.* (of Proposition 4) Let $\mathcal{G}$ be any sub-sigma field of $\mathcal{E}$ that contains $E_\omega$ for all $\omega$. By the Hewitt-Savage 0-1 law [Hewitt and Savage, 1955], $P$ is 0-1 on $\mathcal{E}$ and therefore 0-1 on $\mathcal{G}$. By [Blackwell and Dubins, 1975, Theorem 1], if there exists an 0-1 countably additive probability measure on $\mathcal{G}$ that assigns 0 to all the elements of the finest partition contained in $\mathcal{G}$, then $\mathcal{G}$ is not countably generated. Since $\{E_\omega\}_{\omega\in\Omega}$ is the finest partition contained in $\mathcal{G}$ and $P(E_\omega) = 0$ for all $\omega$, $\mathcal{G}$ is not countably generated. So $\mathcal{E}$ is not quasi-separable. □

## A.2 Theorems

To prove Theorem 3, we use the following lemma.

**Lemma 4.** *Let $\mathcal{F}$ be a sigma-algebra generated by $\{F_i : i \in I\}$. Then for every $A \in \mathcal{F}$, there is a countable $J \subset I$ such that $A \in \sigma\{F_j\}_{j\in J}$.*

*Proof.* Consider the collection

$$\mathcal{F}' = \{A \in \mathcal{F} : \text{there exists countable } J \subseteq I \text{ such that } A \in \sigma\{F_j\}_{j\in J}\}.$$

Clearly $\mathcal{F}' \subseteq \mathcal{F}$ and $F_i \in \mathcal{F}'$ for every $i$. We claim that $\mathcal{F}' = \mathcal{F}$. It suffices to show that $\mathcal{F}'$ is a sigma-algebra.

First, pick any $F_i \in \mathcal{F}$. Then $\Omega \in \sigma\{F_i\}$. So $\Omega \in \mathcal{F}'$.

Next, suppose $A \in \mathcal{F}'$. Then there exists a countable $J \subseteq I$ such that $A \in \sigma\{F_j\}_{j\in J}$. For this same $J$, we have $A^c \in \sigma\{F_j\}_{j\in J}$. So $A^c \in \mathcal{F}'$.

Lastly, suppose $A_n \in \mathcal{F}'$ for $n \geq 1$. Let $J_n$ be the countable sub-collection associated with $A_n$. Let $J = \bigcup_n J_n$. Since $A_n \in \sigma\{F_j\}_{j\in J_n} \subseteq \sigma\{F_j\}_{j\in J}$, we have $\bigcup_n A_n \in \sigma\{F_j\}_{j\in J}$. So $J$ is the countable sub-collection associated with $\bigcup_n A_n$, i.e. $\bigcup_n A_n \in \mathcal{F}'$. This completes the proof that $\mathcal{F}'$ is a sigma-algebra. So $\mathcal{F} \subseteq \mathcal{F}'$ by minimality. □

*Proof.* (of Theorem 3) ($\Rightarrow$) Suppose that $1_{E_\omega}(\nu) \in \mathcal{E} \otimes \mathcal{F}$. Then

$$D = \{(\omega,\nu) \in \Omega \times \Omega : E_\omega = E_\nu\} \in \sigma(\mathcal{E} \times \mathcal{F}).$$

So by the lemma there exist countable collections $\{G_i\}_{i\in I} \subseteq \mathcal{E}$ and $\{F_i\}_{i\in I} \subseteq \mathcal{F}$ such that $D \in \sigma\{G_i \times F_i\}_{i\in I}$. Let $\mathcal{G} = \sigma\{G_i : i \in I\}$. Clearly, $\mathcal{G}$ is countably generated and therefore atomic. Let $G_\omega$ denote the atom of $\mathcal{G}$ containing $\omega$. Moreover, $\mathcal{G} \subseteq \mathcal{E}$. So every $G_\omega$ is $\mathcal{E}$-informed. Since $\omega \in G_\omega$, it follows that $E_\omega \subseteq G_\omega$. We claim that $G_\omega \subseteq E_\omega$ as well, which implies that $G_\omega = E_\omega$ and therefore $E_\omega \in \mathcal{G}$ for all $\omega$.

First, we claim that $E_\omega \times E_\omega \in \mathcal{G} \otimes \mathcal{F}$. Note that $E_\omega \times E_\omega = (E_\omega \times E_\omega) \cap D$. We claim that $(E_\omega \times E_\omega) \cap D = (G_\omega \times E_\omega) \cap D$. Clearly, $(E_\omega \times E_\omega) \cap D \subseteq (G_\omega \times E_\omega) \cap D$. Let $(a,b) \in (G_\omega \times E_\omega) \cap D$. Then $E_a = E_b = E_\omega$. So $a \in E_\omega$ and $(a,b) \in (E_\omega \times E_\omega) \cap D$.

Since $D \in \sigma\{G_i \times F_i\}_{i \in I} \subseteq \mathcal{G} \otimes \mathcal{F}$ and $(G_\omega \times E_\omega) \in \mathcal{G} \otimes \mathcal{F}$, $(G_\omega \times E_\omega) \cap D \in \mathcal{G} \otimes \mathcal{F}$. So $E_\omega \times E_\omega \in \mathcal{G} \otimes \mathcal{F}$.

Next, suppose towards a contradiction that there exists $\omega, \nu$ such that $\nu \in G_\omega$ but $\nu \notin E_\omega$. Note that $\mathcal{G} \otimes \mathcal{F}$ is generated by sets of the form $\{G_i \times F : i \in I, F \in \mathcal{F}\}$. It is a basic fact of measure theory that two elements of $\Omega \times \Omega$ can be separated by an element in the product sigma-field $\mathcal{G} \otimes \mathcal{F}$ if and only if they can be separated by one of its generators. That is, if, for all $i \in I$ and $F \in \mathcal{F}$, $(a, b) \in G_i \times F$ whenever $(c, d) \in G_i \times F$, then, for all $A \in \mathcal{G} \otimes \mathcal{F}$, $(a, b) \in A$ whenever $(c, d) \in A$. Since $\nu \in G_\omega$, the fact above also implies that for every $i$, $\omega \in G_i$ iff $\nu \in G_i$. It follows that for every $a \in \Omega$, $(\omega, a) \in G_i \times F$ iff $(\nu, a) \in G_i \times F$ for any $i$ and $F$. In particular, this holds for $a = \omega$. So $(\omega, \omega)$ and $(\nu, \omega)$ are not separable by any element in $\mathcal{G} \otimes \mathcal{F}$. However, by assumption $\nu \notin E_\omega$. So $(\omega, \omega) \in E_\omega \times E_\omega$ but $(\nu, \omega) \notin E_\omega \times E_\omega$, and $E_\omega \times E_\omega \in \mathcal{G} \otimes \mathcal{F}$, as proved above. A contradiction.

Therefore $E_\omega = G_\omega \in \mathcal{G}$.

($\Leftarrow$) Suppose $\mathcal{E}$ is quasi-separable, and let $G_i$ denote the generators of $G_i \subseteq \mathcal{G}$. Define $G_i^1 = G_i$ and $G_i^0 = \Omega \setminus G_i$. Let

$$\mathcal{G}_n = \{\cap_{i=1}^n G_i^{(c_i)} : c_i \in \{0, 1\}\} \tag{10}$$

We claim that

$$D = \bigcap_{n=1}^{\infty} \left( \bigcup_{G_n \in \mathcal{G}_n} G_n \times G_n \right), \tag{11}$$

where $D$ is defined as in the first part of the proof. As before, since $G_i \in \mathcal{E}$, for every $(\omega, \nu) \in D$ (i.e. $E_\omega = E_\nu$), $\omega \in G_i$ if $\nu \in G_i$. So $D \subseteq \bigcap_i G_i \times G_i$. Now a basic fact in measure theory is that if $\mathcal{G}$ is generated by $\{G_i\}$, then the atoms of $\mathcal{G}$—the elements of the finest partition contained in $\mathcal{G}$—are precisely sets of the form:

$$G_\omega = \{\nu : \forall i, \omega \in G_i \text{ iff } \nu \in G_i\}.$$

By assumption the finest partion contained in $\mathcal{G}$ is $\{E_\omega\}$. So $G_\omega = E_\omega$. Now suppose $(\omega, \nu) \in \bigcap_{n=1}^\infty \left(\bigcup_{G_n \in \mathcal{G}_n} G_n \times G_n\right)$. Then for all $n$, $\nu$ agrees with $\omega$ on its membership to $G_1, \ldots, G_n$. Thus $\nu \in G_\omega$ and hence $E_\nu = E_\omega$, which implies $(\omega, \nu) \in D$.

Since $|\mathcal{G}_n|$ is countable, it follows that $D \in \mathcal{G} \otimes \mathcal{F} \subseteq \mathcal{E} \otimes \mathcal{F}$. Since the graph of the function $(\omega, \nu) \mapsto 1_{E_\omega}(\nu)$ is $D$, it follows that $(\omega, \nu) \mapsto 1_{E_\omega}(\nu)$ is $\mathcal{E} \otimes \mathcal{F}$-measurable. □

*Proof.* (of Theorem 4) Fix a prior probability space $(\Omega, \mathcal{F}, P)$, an evidential situation $\mathcal{E}$ and an updating policy $U$. We assume $U$ satisfies the integral formula (otherwise it is strongly R-dutch bookable, and therefore strongly dutch bookable). Suppose $U$ does not ensure evidential certainty, i.e. the set $G = \{\omega : U_\omega(E_\omega) < 1\}$ has probability greater than 0. Moreover, by assumption, $G \in \mathcal{G}$. Consider the betting strategy $S : \Omega \to \mathcal{X}$ given by

$$S_\omega(\nu) = 1_G(\omega)(U_\omega(E_\omega) - 1_{E_\omega}(\nu)). \tag{12}$$

and the trivial bet $X = 0$. We show that $(X, S)$ is a weak Dutch book against $U$. (It then follows from Proposition 1 that there exists a strong Dutch book against $U$.) Trivially $X$ is an acceptable bet, that is, condition 1 is satisfied. For condition 2, we check that $S$ is a betting strategy and also that the set

$$\{\omega : S_\omega \text{ is acceptable for } U \text{ at } \omega\}$$

is a member of $\mathcal{F}$. Fix $\omega$. If $\omega \in G$, then $S_\omega(\nu) = U_\omega(E_\omega) - 1_{E_\omega}(\nu)$. Since $U_\omega(E_\omega)$ is a constant and $E_\omega \in \mathcal{F}$, $S_\omega$ is $\mathcal{F}$-adapted. If $\omega \notin G$, then $S_\omega = 0$, which is trivially $\mathcal{F}$-adapted. Now fix $\nu$. Note that, since $E$ is partitional, $1_{E_\omega}(\nu) = 1_{E_\nu}(\omega)$. Moreover, $U_\omega(E_\omega) = U_\nu(E_\nu)$. So $S_\omega = U_\omega(E_\omega) - 1_{E_\omega} = U_\nu(E_\nu) - 1_{E_\nu} = S_\nu$. Hence $S$ recommends the same bet at evidentially indistinguishable worlds. Moreover, for each $\omega$, if $\omega \in G$, then

$$\mathbb{E}_{U_\omega}[S_\omega] = \int_\Omega U_\omega(E_\omega) - 1_{E_\omega}(\nu) U_\omega(d\nu) = U_\omega(E_\omega) - U_\omega(E_\omega) = 0 \tag{13}$$

and if $\omega \notin G$, then $S_\omega = 0$ and $\mathbb{E}_\omega[S_\omega] = 0$. So $S$ is acceptable for $U$ at any $\omega$. Hence

$$\{\omega : S_\omega \text{ is acceptable for } U \text{ at } \omega\} = \Omega \in \mathcal{F}.$$

Next we check condition 3. Note that if if $\omega \in G$, then

$$X(\omega) + S_\omega(\omega) = U_\omega(E_\omega) - 1 < 0.$$

Thus $G \subseteq \{\omega : S_\omega \text{ is acceptable for } U \text{ given } \omega \text{ \& } X(\omega) + S_\omega(\omega) < 0\}$, and so

$$P\{\omega : S_\omega \text{ is acceptable for } U \text{ given } \omega \text{ \& } X(\omega) + S_\omega(\omega) < 0\} \geq P(G) > 0,$$

as desired.

Finally we check condition 4. (Note that condition 5 is trivially satisfied since $S$ is acceptable at every $\omega$.) Suppose $\omega \notin G$. Then $X(\omega) + S_\omega(\omega) = 0$. And so

$$P\{\omega : S_\omega \text{ is acceptable for } U \text{ given } \omega \text{ \& } X(\omega) + S_\omega(\omega) > 0\} = P(\emptyset) = 0.$$

Hence $(X, S)$ is a weak dutch book against $U$. □

The proof of the converse result requires three lemmas.

**Lemma 1.** *Let $U$ be an updating policy for $P$ given $\mathcal{E}$ and $S$ a bookie strategy given $\mathcal{E}$. Then the set $A = \{\omega : U_\omega(E_\omega) = 1\}$ belongs to $\mathcal{E}$.*

*Proof.* By the third requirement on an updating policy, the set $A = \{\omega : U_\omega(E_\omega) = 1\} \in \mathcal{F}$, so it suffices to show this set is $E$-informed. Suppose $\omega \in A$, i.e. $U_\omega(E_\omega) = 1$. Let $\nu \in E_\omega$. Then $U_\omega(E_\omega) = U_\nu(E_\omega) = U_\nu(E_\nu) = 1$. Therefore $E_\omega \subset A$ if $\omega \in A$. So $A$ is $\mathcal{E}$-informed. □

**Lemma 2.** *Let $U$ be an updating policy for $P$ given $\mathcal{E}$ and $S$ a bookie strategy given $E$. In what follows, let $\mathbb{E}_\omega$ denote the expectation $\mathbb{E}_{U_\omega}$. If $U$ yields evidential certainty at $\omega$, then $\mathbb{E}_\omega[S_\omega] = \mathbb{E}_\omega[S^*]$. Also, $\{\omega : \mathbb{E}_\omega[S^*] \geq 0\}$ belongs to $\mathcal{E}$.*

*Proof.* Suppose $U_\omega(E_\omega) = 1$. Then

$$\mathbb{E}_\omega[S_\omega] = \int_\Omega S(\omega, \nu) U_w(d\nu) = \int_{E_\omega} S(\omega, \nu) U_\omega(d\nu) = \int_{E_\omega} S(\nu, \nu) U_\omega(d\nu) = \mathbb{E}_\omega[S^*]$$

where the third equality follows from the definition of betting strategies—in particular, $S$ recommends the same bet at any two worlds that are evidentially indistinguishable according to $E$.

For the second claim: by assumption, $U$ does not distinguish possibilities that are evidentially indistinguishable, i.e. $U_\omega = U_\nu$ if $E_\omega = E_\nu$. Thus for all $\nu \in E_\omega$, $\mathbb{E}_\omega[S^*] = \mathbb{E}_\nu[S^*]$. Hence if $\mathbb{E}_\omega[S^*] \geq 0$, then $\mathbb{E}_\nu[S^*] \geq 0$ for all $\nu \in E_\omega$. Therefore the set $\{\omega : \mathbb{E}_\omega[S^*] \geq 0\}$ is $\mathcal{E}$-informed.

We next show that $\{\omega : \mathbb{E}_\omega[S^*] \geq 0\}$ belongs to $\mathcal{F}$. By assumption $S^* \in \mathcal{F}$. First suppose $S^*$ is an indicator $1_A$ with $A \in \mathcal{F}$. Then

$$\mathbb{E}_{(\cdot)}[S^*] = U_{(\cdot)}(A)$$

is $\mathcal{F}$-measurable, since by assumption $U$ is $\mathcal{F}$-adapted, that is, $U[A]$ is $\mathcal{F}$-measurable. If $S^* = \sum_i c_i \cdot 1_{A_i}$, then

$$\mathbb{E}_{(\cdot)}[S^*] = \sum_i c_i U_{(\cdot)}(A_i)$$

which is $\mathcal{F}$-measurable. By a standard exercise we then use the monotone convergence theorem to extend the argument to any $S^* \in \mathcal{F}$. □

**Lemma 3** (Adapted from Çınlar [2011] Proposition IV.2.4.)**.** *Suppose $U$ is an updating policy for $P$ given $E$ which respects the integral formula. Then*

$$U[X] : \omega \mapsto \mathbb{E}_{U_\omega}[X]$$

*satisfies the generalized integral formula, in the sense that for all $Y \in \mathcal{E}$*

$$\mathbb{E}_P[X \cdot Y] = \mathbb{E}_P[U[X] \cdot Y].$$

With these three lemmas in hand, we turn to the proof of the theorem.

*Proof.* (of Theorem 5) Suppose toward contradiction there exists a strong Dutch book $(X, S)$ against $U$. (By Proposition 1, if there exists a weak Dutch book against $U$ then there exists a strong Dutch book against $U$.) Let

$$\Delta_1 = \{\omega : U_\omega(E_\omega) = 1 \ \&\ \mathbb{E}_\omega[S_\omega] \geq 0\}$$
$$\Delta_2 = \{\omega : U_\omega(E_\omega) = 1 \ \&\ \mathbb{E}_\omega[S_\omega] < 0\}.$$

Note that by Lemmas 1 and 3, $\Delta_1, \Delta_2 \in \mathcal{E}$. By evidential certainty, $P(\Delta_1 \cup \Delta_2) = 1$. By assumption, $\mathbb{E}_P[X] \geq 0$. Note also that

$$\mathbb{E}_P[S^* \cdot 1_{\Delta_1}] = \mathbb{E}_P[U[S^*] \cdot 1_{\Delta_1}] = \int_{\Delta_1} \mathbb{E}_\omega[S^*]\, dP(\omega) = \int_{\Delta_1} \mathbb{E}_\omega[S_\omega]\, dP(\omega) \geq 0.$$

Here, the first equality follows from Lemma 3 together with the fact that $\Delta_1 \in \mathcal{E}$ and that $U$ satisfies the integral formula (note that here the countable additivity of $U_\omega$ is crucial). The third equality follows from Lemma 2 and the final inequality follows from the definition of $\Delta_1$. It then follows that

$$\mathbb{E}_P[X + S^* 1_{\Delta_1}] \geq 0. \tag{14}$$

On the other hand,

$$\begin{aligned}\mathbb{E}_P[X + S^* 1_{\Delta_1}] &= \int_\Omega X(\omega)\, dP(\omega) + \int_{\Delta_1} S^*(\omega)\, dP(\omega) \\ &= \int_{\Delta_1} X(\omega)\, dP(\omega) + \int_{\Delta_2} X(\omega)\, dP(\omega) + \int_{\Delta_1} S_\omega(\omega)\, dP(\omega) \\ &= \int_{\Delta_1} (X(\omega) + S_\omega(\omega))\, dP(\omega) + \int_{\Delta_2} X(\omega)\, dP(\omega).\end{aligned}$$

Here the second equality follows from the fact that $P(\Delta_1 \cup \Delta_2) = 1$. Since $(X, S)$ is a strong Dutch book, if $\omega \in \Delta_1$, then $X(\omega) + S_\omega(\omega) < 0$. And if $\omega \in \Delta_2$, then $X(\omega) < 0$. Since either $P(\Delta_1) > 0$ or $P(\Delta_2) > 0$ or both, it follows that

$$\mathbb{E}_P[X + S^* 1_{\Delta_1}] < 0. \tag{15}$$

But this contradicts (14). □

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